On Lanczos' Conformal Trick

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Abstract

The Weyl conformal tensor describes the distorting but volume-preserving tidal effects of gravitation on a material body. A rather complicated combination of the Riemann-Christoffel tensor, the Ricci tensor and the Ricci scalar, the Weyl tensor is used in the construction of a unique conformally-invariant Lagrangian. In 1938 Cornelius Lanczos discovered a clever simplification of the mathematics that eliminated the RC term, thus considerably reducing the complexity of the overall Lagrangian. Here we present an equivalent but simpler approach to the one Lanczos used.

Introduction

In 1918 the German mathematical physicist Hermann Weyl proposed a unification of gravitation and electromagnetism based on the invariance of physics with respect to a conformal (or scale) transformation of the metric tensor $g_{\mu\nu} \to \exp(\pi)\, g_{\mu\nu}$, where $\pi(x)$ is an arbitrary scalar function. A decade later Weyl's idea was recast as *gauge symmetry*, which subsequently became a cornerstone of quantum theory. More recently, the notion of conformal symmetry has been explored in numerous cosmological models, and there is increasing speculation that conformally invariant geometry may indeed underlie Nature.

Weyl's theory, which introduced a non-Riemannian geometry in an effort to embed electromagnetism into general relativity, necessarily relied upon a Lagrangian that was invariant with respect to a local rescaling of the metric tensor. Weyl believed that the scale parameter $\pi(x)$ might be related to the gauge transformation property of electromagnetism via $A_\mu \to A_\mu + \partial_\mu \pi$, and thus provide an opportunity for deriving Maxwell's equations from a purely geometric foundation. The theory failed, but it spurred a considerable amount of interest in gravitational theories based on conformal invariance. That interest has continued to this day, with many researchers contributing to the topic, now properly called *Weyl conformal gravity*.

While he did not employ it in his 1918 theory, Weyl discovered that even in ordinary Riemannian geometry there is a unique tensor quantity that is conformally invariant. Now called the *Weyl conformal tensor* $C^\lambda_{\ \nu\alpha\beta}$, its definition in four dimensions is

$$C^\lambda_{\ \nu\alpha\beta} = R^\lambda_{\ \nu\alpha\beta} + \frac{1}{2}\left(\delta^\lambda_\beta R_{\nu\alpha} - \delta^\lambda_\alpha R_{\nu\beta} + g_{\nu\alpha} R^\lambda_\beta - g_{\nu\beta} R^\lambda_\alpha\right) + \frac{1}{6}\left(\delta^\lambda_\alpha g_{\beta\nu} - \delta^\lambda_\beta g_{\alpha\nu}\right) R \tag{1}$$

where $R^\lambda_{\ \nu\alpha\beta}$ is the Riemann-Christoffel curvature tensor

$$R^\lambda_{\ \nu\alpha\beta} = \begin{Bmatrix}\lambda \\ \nu\alpha\end{Bmatrix}_{|\beta} - \begin{Bmatrix}\lambda \\ \nu\beta\end{Bmatrix}_{|\alpha} + \begin{Bmatrix}\lambda \\ \sigma\beta\end{Bmatrix}\begin{Bmatrix}\sigma \\ \nu\alpha\end{Bmatrix} - \begin{Bmatrix}\lambda \\ \sigma\alpha\end{Bmatrix}\begin{Bmatrix}\sigma \\ \nu\beta\end{Bmatrix}$$

where the terms in braces are the Christoffel (or Levi-Civita) symbols. The the single subscripted bar stands for ordinary partial differentiation; later a double subscripted bar will stand represent covariant differentiation. As can be easily verified, the quantity $C^\lambda_{\ \nu\alpha\beta}$ remains unchanged when the metric tensor is rescaled. Consequently, this tensor was considered early on as a candidate for a generalized version of Einstein's 1915 gravity theory based on this scale or conformal symmetry. The Weyl tensor leads to a unique conformal Lagrangian that can be used to build an alternative gravity theory. That Lagrangian is $\sqrt{-g}\, C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta}$ which, using (1), reduces to

$$\sqrt{-g}\, C_{\mu\nu\alpha\beta}\, C^{\mu\nu\alpha\beta} = \sqrt{-g}\left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2\right) \tag{2}$$

This quadratic quantity is of fourth order with respect to the metric tensor and its derivatives, an undesirable property that greatly complicates the solution of the associated equations of motion. But worse is its mixing of the

Riemann-Christoffel curvature tensor with its Ricci cousins, which complicates consideration of spaces that are Riemann-curved but Ricci-flat (such as the Schwarzschild metric). Nevertheless, if conformal invariance is to be demanded, the Weyl Lagrangian (2) is the only game in town.

An early admirer (and noted investigator) of Weyl's gauge idea was the Hungarian mathematical physicist Cornelius Lanczos, who in a 1938 paper discovered a way to greatly simplify the mathematics by effectively getting rid of the troublesome $R_{\mu\nu\alpha\beta}\,R^{\mu\nu\alpha\beta}$ term in the Lagrangian. We won't reproduce his logic here, but will demonstrate an alternative approach that is equivalent and considerably easier to follow.

Approach

Following Lanczos, we can eliminate the Riemann-Christoffel term $R_{\mu\nu\alpha\beta}\,R^{\mu\nu\alpha\beta}$ if we can find coefficients for the $R_{\mu\nu}R^{\mu\nu}$ and R^2 terms differing from the ones in (2) that leave the integral conformally invariant. We therefore assume that a more general action integral like

$$I = \int \sqrt{-g}\,\left(A R_{\mu\nu\alpha\beta}\,R^{\mu\nu\alpha\beta} + B\,R_{\mu\nu}R^{\mu\nu} + C\,R^2\right) d^4x \qquad (3)$$

exists, where A, B, C are constants.

Now consider the conformal (or scale) change in the metric tensor given by $g_{\mu\nu} \to e^{\pi(x)}\,g_{\mu\nu}$, where $\pi(x)$ is an arbitrary function of the coordinates, while $g^{\mu\nu} \to = -e^{\pi}g^{\mu\nu}$. For an infinitesimal change, we then have the obvious identity $g_{\mu\nu} = (1+e^{\epsilon\pi})\,g_{\mu\nu}$ or $\delta g_{\mu\nu} = \epsilon\pi g_{\mu\nu}$, along with $\delta g^{\mu\nu} = -\epsilon\pi g^{\mu\nu}$, where the constant ϵ is a vanishingly small but finite number.

For the infinitesimal change of scale $\delta g^{\mu\nu} = -\epsilon\pi g^{\mu\nu}$ in four dimensions, the variation of $\sqrt{-g}$ is simple:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu} = 2\,\epsilon\pi\sqrt{-g}$$

The variations for $R_{\mu\nu\alpha\beta}\,R^{\mu\nu\alpha\beta}$, $R_{\mu\nu}R^{\mu\nu}$ and R^2 are considerably more difficult, but the calculations are greatly simplified by using the Palatini identity

$$\delta R^{\lambda}{}_{\nu\alpha\beta} = \left(\delta\begin{Bmatrix}\lambda\\ \nu\alpha\end{Bmatrix}\right)_{||\beta} - \left(\delta\begin{Bmatrix}\lambda\\ \nu\beta\end{Bmatrix}\right)_{||\alpha}$$

(Note that the Christoffel symbols are not tensors, but their variations are).

For simplicity, we will simply write down the variations we'll need. Note that in view of the Palatini identity, we will be using integration by parts numerous times to reduce everything to multipliers of $\delta g^{\mu\nu}$. We'll also assume a local coordinate system, so that the metric tensor and its determinate are constants with respect to partial differentiation. We then have, with the integrals omitted for brevity,

$$\delta\sqrt{-g}\,R_{\mu\nu\alpha\beta}\,R^{\mu\nu\alpha\beta} = 4\,\epsilon\pi\,\sqrt{-g}\,g_{\alpha\beta}R^{\mu\nu}{}_{||\mu||\nu} = 2\,\epsilon\pi\,\sqrt{-g}\,g^{\mu\nu}R_{|\mu||\nu}$$

where we have used the Bianchi identity

$$\left(R^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}R\right)_{||\nu} = 0$$

Similarly,

$$\delta\sqrt{-g}\,R_{\mu\nu}R^{\mu\nu} = 2\,\epsilon\pi\sqrt{-g}\,g^{\mu\nu}R_{|\mu||\nu}$$

along with

$$\delta\sqrt{-g}\,R^2 = 6\,\epsilon\pi\,\sqrt{-g}\,g^{\mu\nu}R_{|\mu||\nu}$$

Variation of (3) thus gives the simple quantity

$$\delta I = \epsilon\,\pi\int\sqrt{-g}\,(2A + 2B + 6C)\,g^{\mu\nu}R_{|\mu||\nu}\,d^4x = 0$$

For conformal invariance the integrand must vanish, so that

$$A + B + 3C = 0 \tag{4}$$

We are free to choose these constants any way we desire, provided they conform to (4). One choice is $A = 1, B = -2, C = 1/3$. But this gives us back (2) again, and so is therefore of no interest. But, like Lanczos, we'd really like to get rid of the Riemann-Christoffel term $R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$, which requires that we set $A = 0$. Then from (4) we can take $B = 1, C = -1/3$. This then gives the conventional identity for the conformal gravity action

$$I = \int \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) d^4x \tag{5}$$

Let us now see what Lanczos did. Following a completely different logic, he effectively chose $A = 1, B = -4, C = 1$, giving

$$I_L = \int \sqrt{-g} \left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) d^4x \tag{6}$$

This is still conformally invariant, but it still has the Riemann-Christoffel term. So Lanczos subtracted this quantity from (2), getting rid of the Riemann-Christoffel term and arriving at

$$I - I_L = 2 \int \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) d^4x$$

which, ignoring the meaningless factor of 2, is identical to (5). In view of this, Lanczos identified the quantity I_L as a pure *divergence term*, **an erroneous identification that has persisted for nearly ninety years in the literature**. Note that *any* set of constants conforming to (4) leads to a conformally invariant action. We could have chosen $A = -17.2, B = 9.7, C = 2.5$ instead, and adding the resulting action to (2) multiplied by 17.2 would have given (5) multiplied by a meaningless constant. In other words, (6) is just a variational constant, as is (2) and (5), and their differences are also variational constants.

Conformal Gravity Theory

An arbitrary variation of the metric tensor in (2) leads to

$$\delta \int \sqrt{-g}\, C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} d^4x = \int \sqrt{-g}\, W_{\mu\nu} \delta g^{\mu\nu} d^4x$$

where $W_{\mu\nu}$ is a complicated expression involving the Ricci tensor and scalar and their derivatives (in a space where matter is present, $W_{\mu\nu}$ would be proportional to the symmetric energy tensor $T_{\mu\nu}$). An exact solution for the vacuum case $W_{\mu\nu} = 0$ has been worked out in great detail by Mannheim and Kazanas using the Schwarzschild line element

$$ds^2 = e^{\nu}(dx^0)^2 - e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2$$

The solution they found is

$$e^{\nu} = e^{-\lambda} = 1 - \frac{\beta(2 - 3\gamma)}{r} - 3\beta\gamma + \gamma r - k r^2$$

where β, γ and k are arbitrary constants. The resemblance of the Mannheim-Kazanas solution to the ordinary Schwarzschild solution is obvious. The linear and quadratic terms in r very possibly have application to the solution of the galactic rotation, dark matter and dark energy problems.

References

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